Intrinsic linewidth of a free-electron laser with an axial-guide magnetic field

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The intrinsic linewidth of a free-electron laser with an axial-guide magnetic field is analyzed using a master Fokker-Planck equation. All electron emissions caused by the axial bremsstrahlung and the transverse cyclotron have contributions to the linewidth.

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Some previous works [1–3] have discussed the problem of the intrinsic linewidth of a wiggler-pumped free-electron laser. In contrast to the quantum-mechanical characteristic of the linewidth in the ordinary, i.e., atomic or molecular, lasers, the linewidth in free-electron laser has a classical expression, although intrinsic linewidths were determined by spontaneous emissions in both ordinary and free-electron lasers. The reason for this difference is that the spontaneous emissions in ordinary lasers are caused by quantum transitions among energy levels of atoms, but emissions in the free-electron laser are caused by the classical bremsstrahlung in the wiggler field.

In general, an axial-guide magnetic field is usually introduced in free-electron lasers for beam collimation. The guide field also results in the transverse cyclotron of the electron beam. In this case, both the axial bremsstrahlung in the wiggler field and the transverse cyclotron in the guide field of the electron beam emit laser photons spontaneously. And the two kinds of spontaneous emissions have contributions to the intrinsic linewidth of a free-electron laser.

The goal of this Brief Report is to analyze the intrinsic linewidth of a wiggler-pumped free-electron laser with an axial-guide magnetic field from a master Fokker-Planck equation. The procedure is similar to that of Ref. [3]. We start from the three-dimensional single-particle Bambini-Renieri Hamiltonian of a free-electron laser with an axial-guide magnetic field [4]

$$H=mc^2+\hbar\omega+\tfrac{1}{2}\hbar|\omega_c|+\hbar\omega[a_W^\dagger a_W+a_L^\dagger a_L]+\hbar|\omega_c|T_+T_-+\frac{\hat{p}_z^2}{2m}+\hbar\Omega[a_W^\dagger a_L e^{2ikz}+a_W a_L^\dagger e^{-2ikz}]$$
$$-\frac{\hbar}{2}[|\omega_c|\Omega]^{1/2}\left\{a_W^\dagger\left[\left(1-\frac{\omega_c}{|\omega_c|}\right)T_+ +\left(1+\frac{\omega_c}{|\omega_c|}\right)T_-\right]e^{ik_1z}+a_W\left[\left(1+\frac{\omega_c}{|\omega_c|}\right)T_+ +\left(1-\frac{\omega_c}{|\omega_c|}\right)T_-\right]e^{-ik_1z}\right.$$
$$+a_L^\dagger\left[\left(1-\frac{\omega_c}{|\omega_c|}\right)T_+ +\left(1+\frac{\omega_c}{|\omega_c|}\right)T_-\right]e^{-ik_2z}$$
$$\left.+a_L\left[\left(1+\frac{\omega_c}{|\omega_c|}\right)T_+ +\left(1-\frac{\omega_c}{|\omega_c|}\right)T_-\right]e^{ik_2z}\right\}, \quad (1)$$

where $\hbar$ is Planck constant, c is the speed of light in vacuum, m is the electron mass, $\mathbf{r}=(x,y,z)$ is the electron position, $\hat{\mathbf{p}}=(\hat{p}_x,\hat{p}_y,\hat{p}_z)$ is the linear momentum of electron, B_0 is the strength of the guide field, $\omega_c=eB_0/m$ is the cyclotron frequency, Ω is the coupling constant, ω is the frequency of the wiggler and laser fields in Bambini-Renieri frame, k_1 (k_2) is the wave number of the wiggler (laser) field, $k=(k_1+k_2)/2$, a_W (a_L) and $a_W^\dagger$ ($a_L^\dagger$) are the annihilation and creation operators of the wiggler (laser) field, and

$$T_\pm=\frac{\left(\hat{p}_x+\dfrac{m\omega_c}{2}y\right)\mp\dfrac{i\omega_c}{|\omega_c|}\left(\hat{p}_y-\dfrac{m\omega_c}{2}x\right)}{\sqrt{2m\hbar|\omega_c|}} \quad (2)$$

are the annihilation and creation operators of the electron transverse motion with the commutator $[T_-,T_+]=1$. For conventional free-electron laser devices, the wiggler field is much stronger than the laser field, so we can treat it as a c-number field, that is $a_W,a_W^\dagger\sim\sqrt{N_W}$. Then, in the interaction picture, the interaction Hamiltonian is written as

$$H_I=\hbar\Omega\sqrt{N_W}\left[a_L e^{2ikz}e^{i\alpha_{(\hat p_z)}t}+a_L^\dagger e^{-i\alpha_{(\hat p_z)}t}e^{-2ikz}\right]$$
$$-\frac{\hbar}{2}\sqrt{N_W|\omega_c|\Omega}\left[\left[1-\frac{\omega_c}{|\omega_c|}\right]T_+e^{ik_1z}e^{i\beta_{+(\hat p_z)}t}+\left[1-\frac{\omega_c}{|\omega_c|}\right]T_-e^{-i\beta_{+(\hat p_z)}t}e^{-ik_1z}\right.$$
$$\left.+\left[1+\frac{\omega_c}{|\omega_c|}\right]T_+e^{-i\beta_{-(\hat p_z)}t}e^{-ik_1z}+\left[1+\frac{\omega_c}{|\omega_c|}\right]T_-e^{ik_1z}e^{i\beta_{-(\hat p_z)}t}\right]$$
$$-\frac{\hbar}{2}\sqrt{|\omega_c|\Omega}\left[\left[1+\frac{\omega_c}{|\omega_c|}\right]a_LT_+e^{ik_2z}e^{i\gamma_{+(\hat p_z)}t}+\left[1+\frac{\omega_c}{|\omega_c|}\right]a_L^\dagger T_-e^{-i\gamma_{+(\hat p_z)}t}e^{-ik_2z}\right.$$
$$\left.+\left[1-\frac{\omega_c}{|\omega_c|}\right]a_LT_-e^{ik_2z}e^{i\gamma_{-(\hat p_z)}t}+\left[1-\frac{\omega_c}{|\omega_c|}\right]a_L^\dagger T_+e^{-i\gamma_{-(\hat p_z)}t}e^{-ik_2z}\right], \tag{3}$$

where

$$\alpha_{(\hat p_z)}=\frac{2\hbar k^2+2k\hat p_z}{m}, \tag{4}$$

$$\beta_{\pm(\hat p_z)}=\omega\pm|\omega_c|+\frac{\hbar k_1^2+2k_1\hat p_z}{2m}, \tag{5}$$

$$\gamma_{\pm(\hat p_z)}=-\omega\pm|\omega_c|+\frac{\hbar k_2^2+2k_2|\hat p_z}{2m}. \tag{6}$$

The density operator $\rho_{e\text{-}l}$ of the electron-laser system obeys Liouville's equation [5]

$$i\hbar\frac{\partial\rho_{e\text{-}l}(t)}{\partial t}=[H_I(t),\rho_{e\text{-}l}(t)]. \tag{7}$$

To further the discussion, we assume that the initial electron-laser field density operator can be written as a product of the electron, which is in the eigenstates $|p_0\rangle$ of the axial momentum $\hat p_z$ with the eigenvalue p_0 and $|N_{T0}\rangle$ of the transverse motion operator T_+T_- with the eigenvalue N_{T0}, and the laser field, which is in arbitrary mixed or pure state. That is,

$$\rho_{e\text{-}l}=\sum_{n_{L_1},n_{L_2}}\rho_{n_{L_1},n_{L_2}}|n_{L_1}\rangle\langle n_{L_2}|\otimes|p_0\rangle\langle p_0||N_{T0}\rangle\langle N_{T0}|. \tag{8}$$

Solving (7) through iteration and taking the trace Tr_e over the electron operators, one can obtain the reduced density operator $\rho=\mathrm{Tr}_e\rho_{e\text{-}l}$ of the laser field

$$\rho_{(t+T)}=\rho_{(t)}-\frac{1}{\hbar^2}\int_t^{t+T}dt_1\int_t^{t_1}dt_2\mathrm{Tr}_e[H_I(t_1),[H_I(t_2),\rho_{e\text{-}l}(t)]]. \tag{9}$$

After some tedious manipulations, the "coarse-grained" time rate of change [5] for $\rho(t)$ is then given by

$$\frac{d}{dt}\rho(t)=r_e[\rho(t+T)-\rho(t)]$$
$$=\left\{r_eN_W\Omega^2I[\alpha(p_0-2\hbar k)]+r_e|\omega_c|\Omega\left[N_{T0}+\frac{(|\omega_c|-\omega_c)}{2|\omega_c|}\right]I[\gamma(p_0-\hbar k_2)]\right\}[a_L^\dagger\rho(t)a_L-a_La_L^\dagger\rho(t)]$$
$$+\left\{r_eN_W\Omega^2I[\alpha(p_0)]+r_e|\omega_c|\Omega\left[N_{T0}+\frac{(|\omega_c|+\omega_c)}{2|\omega_c|}\right]I[\gamma(p_0)]\right\}[a_L\rho(t)a_L^+-\rho(t)a_L^\dagger a_L]+\text{H.c.}, \tag{10}$$

where r_e is the rate of injection of electrons per second, and

$$\gamma_{(\hat p_z)}=\omega_c+\frac{[\gamma_{+(\hat p_z)}+\gamma_{-(\hat p_z)}]}{2}, \tag{11}$$

$$I[\lambda]=\int_t^{t+T}dt_1\int_t^{t_1}dt_2e^{i\lambda(t_1-t_2)}. \tag{12}$$

Substituting the coherent-state representation

$$\rho(t)=\int\int d^2\alpha\,P(\alpha,\alpha^*,t)|\alpha\rangle\langle\alpha| \tag{13}$$

into (10) yields

$$\frac{\partial}{\partial t}P(\alpha,\alpha^*,t)=\left\{r_e N_W\Omega^2 I[\alpha(p_0-2\hbar k)]+r_e|\omega_c|\Omega\left[N_{T0}+\frac{(|\omega_c|-\omega_c)}{2|\omega_c|}\right]I[\gamma(p_0-\hbar k_2)]\right\}\frac{\partial^2 P(\alpha,\alpha^*,t)}{\partial\alpha\partial\alpha^*}$$
$$+\left\{r_e N_W\Omega^2\{I[\alpha(p_0)]-I[\alpha(p_0-2\hbar k)]\}+r_e|\omega_c|\Omega\left[N_{T0}+\frac{(|\omega_c|+\omega_c)}{2|\omega_c|}\right][\gamma(p_0)]\right.$$
$$\left.-r_e|\omega_c|\Omega\left[N_{T0}+\frac{(|\omega_c|-\omega_c)}{2|\omega_c|}\right]]I[\gamma(p_0-\hbar k_2)]\right\}\frac{\partial}{\partial\alpha^*}[\alpha^* P(\alpha,\alpha^*,t)]+\text{H.c.} \quad (14)$$

Under the steady-state operating condition, we can set the laser intensity equal to the steady-state average value $\bar{n}_{SS}$, and then obtain the decay constant from the above formula,

$$D=\frac{r_e N_W\Omega^2\text{Re}I[\alpha(p_0-2\hbar k)]+r_e|\omega_c|\Omega\left[N_{T0}+\frac{(|\omega_c|-\omega_c)}{2|\omega_c|}\right]\text{Re}I[\gamma(p_0-\hbar k_2)]}{\bar{n}_{SS}} . \quad (15)$$

The ultimate linewidth is a classical result (independent of $\hbar$). In the guide-field-free case, (15) drives at the previous conclusions [1–3]. In the wiggler-free case, (15) gives the decay rate of free-electron laser only with the guide field. In the general case, both the wiggler and the guide field have contributions to the linewidth. By using the following definitions:

$$\theta_1=\frac{kp_0T}{m},\quad \theta_2=\frac{(\omega-\omega_c)T}{2},\quad \nu=\frac{\hbar k^2}{2m},\quad \text{sinc}(q)=\left[\frac{\sin q}{q}\right]^2 ,$$

one can write (15) in an explicit expression,

$$D=\frac{r_e N_W\Omega^2T^2\text{sinc}(\theta_1-2\nu T)+r_e\Omega T^2[N_{T0}|\omega_c|+\frac{1}{2}(|\omega_c|-\omega_c)]\text{sinc}\left[\theta_2-\frac{k_2\theta_1}{2k}+\frac{k_2^2\nu T}{2k^2}\right]}{2\bar{n}_{SS}} . \quad (16)$$

For the extreme relativistic case $V_{z0}\lesssim c$, where V_{z0} is the axial velocity of the electron beam in the laboratory frame, we have $k_1=k_2=k$ and $\omega=ck$. Then, in the limit θ_1 and $\theta_2 \gg \nu T$, (16) drives at

$$D=\frac{r_e N_W\Omega^2T^2}{2\bar{n}_{SS}}\left[\text{sinc}(\theta_1)-2\nu T\frac{d}{d\theta_1}\text{sinc}(\theta_1)\right]+\frac{r_e\Omega T^2[N_{T0}|\omega_c|+\frac{1}{2}(|\omega_c|-\omega_c)]}{2\bar{n}_{SS}}\times\left[\text{sinc}(\theta_2-\tfrac{1}{2}\theta_1)+\frac{\nu T}{2}\frac{d}{d\theta_2}\text{sinc}(\theta_2-\tfrac{1}{2}\theta_1)\right] , \quad (17)$$

where $G_1\propto-(d/d\theta_1)\text{sinc}(\theta_1)$ is the gain [6] of the guide-field-free free-electron laser and $G_2\propto(d/d\theta_2)\text{sinc}(\theta_2-\frac{1}{2}\theta_1)$ is the gain [7] of the wiggler-free free-electron laser. The maximum gains occur at $\theta_1=1.3$ and $\theta_2-\frac{1}{2}\theta_1=-1.3$, and the corresponding ultimate line-width of the wiggler-pumped free-electron laser with an axial-guide magnetic field is

$$D\cong\frac{r_e N_W\Omega^2T^2}{2\bar{n}_{SS}}\times\left[1+\frac{N_{T0}|\omega_c|+\frac{1}{2}(|\omega_c|-\omega_c)}{N_W\Omega}\right]\text{sinc}(1.3) . \quad (18)$$

So the broadening effect of transverse motion of electron on the linewidth of a free-electron laser is described by the parameter

$$\xi=[N_{T0}|\omega_c|+\tfrac{1}{2}(|\omega_c|-\omega_c)]/N_W\Omega . \quad (19)$$

In laboratory-frame variables, we have

$$N_W\Omega=\frac{e^2B_W^2}{2m\hbar k_W^2} , \quad (20)$$

where k_W (B_W) is the wave number (strength) of the wiggler field. Further, we let the initial energy $\hbar|\omega_c|[N_{T0}+(|\omega_c|-\omega_c)/2|\omega_c|]$ be the nonrelativistic classical equilibrium energy of the vertical motion of electron, approximately; then (19) gives

$$\xi=\frac{\left[\frac{\mathbf{v}_\perp}{c}\right]^2}{\left[\frac{eB_W}{mck_W}\right]^2} . \quad (21)$$

For conventional experimental parameters, the perpendicular component $\mathbf{V}_\perp$ of the electron velocity should be small, about 10% of c, and $(eB_W/mck_W)\sim5\times10^{-2}$; then the above formula gives

$$\xi\sim4 . \quad (22)$$

The transverse motion of electron will enhance the linewidth of a free-electron laser several times.

Finally, it is necessary to point out that the Lorentz equation governing the classical electron motion is invariant under the transformation $\mathbf{A}_r \rightarrow -\mathbf{A}_r$, $\mathbf{A}_W \rightarrow -\mathbf{A}_W$, $\mathbf{A}_0 \rightarrow -\mathbf{A}_0$, and $t \rightarrow -t$, where $\mathbf{A}_r$, $\mathbf{A}_W$, and $\mathbf{A}_0$ are the vector potentials of the radiation, wiggler, and axial-guide magnetic fields, respectively. Correspondingly, the interaction Hamiltonian in quantum mechanics is also invariant under the same transformation. However, if we perform the transformation $\mathbf{A}_0 \rightarrow -\mathbf{A}_0$ (the guide field is inverted) only, then the time-reversal invariance is invalid and the role of T_+ and T_- in the interaction Hamiltonian is replaced by that of T_- and T_+, respectively. It is this feature that causes the intrinsic linewidth D to depend on the sign of ω_c.

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